SELECTED ECONOMIC APPLICATIONS OF ENTROPY

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1. Introduction

For many years, science development has consisted of deepening specialization, reductionism and analytical approach. Nowadays, more and more scientists are aware of the need to apply holistic, interdisciplinary or evolutionary approach. One of manifestations of this trend is recent interest in the category of entropy in the field of economics which results from the limits of classical economics based on the methodological individualism.

Obviously, owing to the heterogeneity of the category of entropy, representing most often disorder, probability, lack of information or the extent of degradation of energy, its applications in economic theory are also diverse. In principle, there are no complex and exhaustive studies basing on the energetic or monetary aspects of entropy. Most of economic interpretations refer to Boltzmann-Shannon entropy and related measures. However, thermodynamic approach seems to be more promising in this discipline.

The aim of this study is to present and to order existing applications of entropy related to economics and to point out at new directions of its possible applications.

2. Entropic measures

Entropic measures (presented in table 1) replenish classic measures and measures of position and may find application not only in economics but also in other disciplines.

<table>
<thead>
<tr>
<th>Measure</th>
<th>Formula</th>
<th>Comments</th>
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<tr>
<td>empirical (real) entropy</td>
<td>$H = -\sum_{i=1}^{n} p_i \log_2 p_i$ where $p_i$ is probability of appearance of variant $i$</td>
<td>$H$ depends only on probability. $H$ is always nonnegative. $H = 0$ when probability of one variant is 1. $H = H_{\text{max}}$ when all probabilities are equal.</td>
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The results in the paper were presented on the XIII-th Conference on Mathematics and Computer Science, Chełm, July 1–4, 2007.
### Maximum Entropy

<table>
<thead>
<tr>
<th>Maximum Entropy</th>
<th>( H_{\text{max}} = \log_2 n ), where ( n ) is the number of variants of the feature.</th>
<th>( H_{\text{max}} ) is the upper limit of empirical entropy.</th>
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### Index of Entropy (IH)

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<tr>
<th>Index of Entropy (IH)</th>
<th>( IH = \frac{H}{H_{\text{max}}} )</th>
<th>( IH ) and ( WH ) allow comparing entropies of distributions when the numbers of variants of features in these distributions are not equal. Maximum diversity occurs for ( IH = 1, WH = 0 ). Maximum concentration occurs for ( IH = 0, WH = 1 ).</th>
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<tr>
<td>Coefficient of Entropy (WH) (Redundancy)</td>
<td>( WH = 1 - \frac{H}{H_{\text{max}}} )</td>
<td>( w_ih ) are useful to analyze graphically the distributions and to calculate entropic measure of asymmetry.</td>
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### Partial Coefficients of Entropy

| Partial Coefficients of Entropy | \( w_ih = \frac{|p_i \log_2 p_i|}{H_{\text{max}}} \) | \( w_ih \) are useful to analyze graphically the distributions and to calculate entropic measure of asymmetry. |
|---------------------------------|-------------------------------------------------|-------------------------------------------------|

### Entropic Measure of Asymmetry

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<tr>
<th>Entropic Measure of Asymmetry</th>
<th>1) ( HAs = \sum_{i=1}^{k-1} w_ih - \sum_{i=k+1}^{n} w_ih ), where ( k ) is the number of the middle class (for statistical series with odd number of classes)</th>
<th>( HAs ) finds application especially in case of multimodal series, when measures of position lead to false conclusions.</th>
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<td>2) ( HAs = \sum_{i=1}^{k_1} w_ih - \sum_{i=k_2}^{n} w_ih ), where ( k_1 ) and ( k_2 ) is the first and the second middle class (for statistical series with even number of classes)</td>
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### Entropic Coefficient of Transformation

<table>
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<th>( WTH = \frac{H(t+k)}{H_{\text{max}}(t+k)} - \frac{H(t)}{H_{\text{max}}(t)} )</th>
<th>( WTH ) characterizes the similarity of given structure in time ( t+k ) compared to time ( t ). ( WTH = 0 ) for maximum reflection of structure diversity from time ( t ) in time ( t+k ). ( WTH = 1 ) when maximum concentration occurs in time ( t ) and maximum evenness in time ( t+k ). ( WTH = -1 ) when maximum evenness occurs in time ( t ) and maximum concentration in time ( t+k ). ( WTH &gt; 0 ) for decrease of concentration. ( WTH &lt; 0 ) for increase of concentration.</th>
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In many cases entropic measures gain an advantage over traditional measures as they make it possible to analyze more than one feature of a given phenomenon and to compare combinations of quantitative, qualitative or quantitative and qualitative features.

3. Entropy as the measure of concentration, inequalities, convergence, transparency and propensities

In the economic literature entropy is treated most often as the measure of concentration of the companies functioning on different markets, illustrating the extent to which the market is dominated by the biggest companies. The interest in concentration of different market structures is crucial both for the theory of economy and economic law.

The entropy measure of concentration is being used apart from other concentration measures, such as four or eight biggest companies concentration ratio. The entropy measure of concentration weights the share of each company \((P_i)\) by the logarithm of \(\frac{1}{P_i}\), and follows the formula:

\[
E = \sum_{i=1}^{n} P_i \log_2 \frac{1}{P_i}
\]

or equivalently

\[
E = -\sum_{i=1}^{n} P_i \log_2 P_i.
\]

One of the advantages of this measure is sensitivity with reference to small companies ([3] p.360). It is also additively decomposable in contrast to other concentration measures ([4] p.141).

For instance, on the basis of *Fortune’s* rankings of the 500 largest companies from the United States of America (taking into account their structural changes) it can be stated that in this group of companies a statistically significant increase in concentration in terms of sales has occurred ([4] and author’s calculations based on *Fortune’s* directories from 1955-2007).

Analogically, entropic measures reflect income inequalities in the society and to some extent they can be an alternative for the well-known Gini index [5].

Entropy finds application in the analysis of economic convergence which is related most often to the levels of per capita GDP in different regions or countries. A test for \(\sigma\)-convergence is a test for the presence of a decreasing trend in an inequality measure, i.e. standard deviation, Gini index or entropy ([6] p.500). However, limited data availability (short time series) and structural changes in GDP make it difficult to draw binding conclusions referring to economic convergence.

Entropic measures may be applied to assess the markets or stock exchanges, that is in the process of investment decision making. Lower entropy (calculated on the basis of the total return, price-earning ratio or other indices) shows greater determination and greater transparency of the markets (stock exchanges) and increases the probability of right forecasts and decisions [7].

The purposefulness (teleology) of human activity makes the propensities play a fundamental role in the economic process. The most significant, such as marginal propensity to consume, save or import, have been analyzed since the time of works of J.M. Keynes. However, one can analyze also human propensities to risk, migrate, smoke cigarettes,
drink alcohol, innovate, work hard or others, which have high impact on economic process as well. When the choice is limited to two variants ($A$ and $A' = 1 - A$) the entropy measure of propensity can be calculated as follows:

$$H(A, 1 - A) = -A \log_2 A - (1 - A) \log_2 (1 - A).$$

Lower entropy means that the socio-economic system is more determinated and more predictable ([8] p.39).

In all the cases mentioned above lower entropy implies higher concentration, greater inequalities, higher level of economic convergence, determination, transparency or stronger propensities.

Occurrence of maximum entropy in a given system is perceived as a symptom of the equilibrium state of the system. This implies growing interest in the application of the maximum entropy principle.

For instance, the maximum entropy principle can be applied in the analysis of utility function and risk aversion [9] (one has to be aware of the difference between utility related to maximum entropy and entropy maximizing utility, the so-called u-entropy [10]).

Another example is application of the maximum entropy principle in key sector analysis basing on a decomposition of the input-output matrix, which allows identifying the most probable direction of economic development in a given country [11].

Besides, the maximum entropy principle in the field of economics has not led to many applications so far.

4. Entropy and money distribution

Similarly to the law of energy conservation, in a closed economic system, where the money supply ($M$) is constant, economic agents have no savings nor debts, the equilibrium probability distribution of money possessed by these agents follows Boltzmann-Gibbs law:

$$P(m) = Ce^{-m/T},$$

where: $m$ - money (equivalent of energy), $T$ - average amount of money per agent (equivalent of temperature), $C$ - normalizing constant.

This distribution results from additivity of money ($m = m_1 + m_2$) and multiplicativity of probabilities ($P = P_1 P_2$).

$$P(m_1 + m_2) = P(m_1)P(m_2),$$

so $P(m) = Ce^{-m/T}$, where $C = 1/T$ and $T = M/N$ ($N$ is the number of agents) [12].

The length of time for the system to reach the equilibrium depends on the amount of money exchanged. The bigger the amount, the faster the increase of entropy and the faster the system reaches the equilibrium state.

The law of money conservation reflects the fact that money cannot be created by economic agents but only transferred. It is worth emphasizing that money distribution cannot be identified with wealth distribution. The latter is a broader concept including material product not subject to the conservation law because of their being manufactured, consumed or destroyed.

This model allows to calculate the optimal price that is maximizing monopolist’s profit ($p^\ast = T$). Furthermore, it explains the mechanism of exchange between two
disconnected systems - when \( T_1 \neq T_2 \) (this difference implies non-equilibrium state stimulating speculations) the two systems are functioning together like heat engine [12].

If the agents are permitted to go into debt, the distribution of money also follows Boltzmann-Gibbs law, but \( T = M/N + m_d \) (where \( m_d \) stands for the limit of the debt of an agent), which means greater inequalities between the agents [12].

However, there is no Boltzmann-Gibbs distribution when the agents make an effort to save some money. Introduction of propensity to save is justified by the fact that in reality no agent trades with entire amount of money he possess [13].

5. Entropy, innovations and motivation

Many economic applications of entropy still wait for their quantitative interpretation. One of very interesting examples is the relation between the entropy and innovations, especially in the context of knowledge-based economy.

The main source of innovation is freely diffusing information. But paradoxically, the faster is the pace of innovation, the faster erosion of existing competitive advantage resulting from innovation ([14] p.5). This erosion (loss occurring because of voluntary or involuntary sharing of information) can be just illustrated by the entropy. Therefore value of a given innovation depends mostly on the pace of its diffusion through the system. An innovation that diffuses quickly provides a short-lived competitive advantage, although potentially its value could be greater. Hence, there are many attempts to protect the competitive advantage by legal means of intellectual property protection. Besides, one can distinguish innovations of low entropy (providing long-lasting competitive advantage) and innovations of high entropy (diffusing quickly) ([14] p.21-23).

Another example is the effort entropy related to the problem of motivation, productivity and utility of effort. The effort entropy appears when the productivity of people, according to which a given person is deciding on his personal effort standard, is decreasing. As a result the role of standards and goals in the organization is diminishing ([15] p.336).

6. Entropy and mergers

Analogically to the endoenergetic smelting alloys of metals, unification of economic systems or merger of companies requires certain expenditures. It is most convenient to analyze entropy, inner energy and free energy of mixing in the case of binary systems [16]. As far as entropy of a merger is concerned one has to pay attention to the irreversibility of this process and to realize that total entropy of the merged system cannot be calculated as simply as

\[ s_{x,id} = (1 - x)s_{A,id} + xs_{B,id}, \]

(where \( s_{A,id} \) and \( s_{B,id} \) are entropies of system \( A \) and \( B \), \( x \) and \( 1 - x \) are portions of system \( A \) and \( B \) in the compound system; see the dotted line on figure 1).

In the ideal case the curve of increase of entropy is given by the following formula:

\[ \Delta_M s_{x,id} = -R[(1 - x) \ln(1 - x) + x \ln x], \]

where \( R \) is constant. This curve is symmetrical between points \( s_{A,id} \) and \( s_{B,id} \) and has maximum when \( x = 1 - x = 0.5 \) (see the bold curve on figure 1). In reality the merger requires another expenditure (entropy supply) depending on the real inner energy of mixing (see the top curve on figure 1) ([16] p.82).
The entropic balance of every merger or integration is negative. Furthermore, the additional entropic burden (loss of energy) may be greater for the system characterized by lower entropy before merger (system A on figure 1). This raises many questions about entropic limits or consequences of mergers and integration ([16] p.83). This issue seems to be crucial especially in the context of globalization, growing importance and value of cross-border mergers and acquisitions in the inflow of foreign direct investment all over the world, as well as many dilemmas related to costs and benefits from European integration.

**Final remarks**

In conclusion, despite its universality, the category of entropy has not lived to exhaustive economic interpretations so far. There is no coherent entropic theory of economics and it is not certain that if such a general theory comes into being, it will be better than existing ones in terms of its explanatory potential and applicability. However, more detailed theories (like entropic theory of innovation or entropic theory of globalization) are surely worth formulating. Against this background, one can outline the possibility to analyze in terms of entropy complex non-linear economic relations in different scales and different time. Such an analysis could focus on characterization of monopolization processes on different markets, process of globalization, comparison of growth and development in different countries and their eventual limits, convergence paths (e.g. in
the European Union), system’s stability, economies of scale, taxation, monetary policy and other socio-economic issues.

References